Bitcoin Econometrics

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Overview of the Presentation

- What is bitcoin's fundamental value? A review of financial and economic approaches
- Modelling bitcoin price dynamics
- Market price discovery
- Detecting bubbles and explosive behaviour in bitcoin prices
- Conclusions

What is bitcoin fundamental value? A review of financial and economic approaches

A long-term upper bound: *Market Sizing*

Market sizing is basically the process of estimating the potential of a market and this is widely used by companies which intend to launch a new product or service.

Woo et al. (2013) in a Bank of America Merrill Lynch report estimated separately the value of bitcoin as a A) *medium of exchange* and as B) *store of value* and then summed them up to get a rough estimate of bitcoin fair value.

What is bitcoin fundamental value? A review of financial and economic approaches

1. More specifically, to compute the value as medium of exchange, they considered two uses for bitcoin: *e-commerce* and *money transfer*:

$$V_{e-commerce_t} = \frac{1}{10}\left(\sum_{i=1}^{10} \frac{HD_{US_{t-i}}}{C_{US_{t-i}}}\right) \cdot \mathrm{B2C}_{\mathrm{t\text{-}1}} \cdot Bitcoin_{share} \cdot \frac{GDP_{world_{t-1}}}{GDP_{US_{t-1}}}$$

which is approximately \$5bn worth of Bitcoins for the total global on-line shopping.

$$V_{money\ transfer_t} = \frac{1}{3}\left(MK_{WU_t} + MK_{MG_t} + MK_{E_t}\right)$$

that is, the the average market capitalization of Western Union, MoneyGram and Euronet (approximately \$ 4.5bn)

What is bitcoin fundamental value? A review of financial and economic approaches

2. Woo et al. (2013) suggested that the closest assets to bitcoin as a *store value* are probably precious metals or cash.

They suggested that the Bitcoin market capitalization for its role as a store of value could reach \$5bn.

Interestingly, they noted that this value is close to the value of the total US silver eagles minted since 1986 (around \$8bn - 12k tons)

$$V_{store\ of\ value_t} = 0.6 \cdot TSM_t \cdot P_{silver,t}$$

where TSM_t is the total sum of all US silver eagles minted since 1986 at time t, while $P_{silver,t}$ is the price for 1 troy ounce of silver at time t.

What is bitcoin fundamental value? A review of financial and economic approaches

Finally, Woo et al. (2013) computed the potential bitcoin fair value as

$$P_{bitcoin_t} = \frac{(V_{e-commerce_t} + V_{money\ transfer_t} + V_{store\ of\ value_t})}{TB_t}$$

where TB_t is the total number of bitcoin in circulation, thus obtaining a maximum fair value of Bitcoin approximately equal to 1300$ (**Woo et al. (2013) used data up to 2012**).

Finally, a similar approach is investigated by Huhtinen (2014), who considered the current money aggregates M2 for USD, EUR and JPY, and alternative scenarios for the portion of money supply that could be replaced by bitcoin, instead. He argues the most realistic replacement level is 0.1{%} and it could be achieved with a bitcoin valuation of 1573 euro.

A short-term lower bound: *the marginal cost of bitcoin production*

Garcia et al. (2014) were the first to suggest that the fundamental value of one bitcoin should be at least equal to the cost of the energy involved in its production through mining.

$\Rightarrow$ lower bound estimate of bitcoin fundamental value.

More recently, a more refined model for the cost of bitcoin production was developed by Hayes (2015a,b). Variables to consider:

1. the cost of electricity, measured in cents per kilowatt-hour;
2. the energy consumption per unit of mining effort, measured in watts per GH/s (1 W/GH/s=1 Joule/GH);
3. the bitcoin market price;
4. the difficulty of the bitcoin algorithm;
5. the block reward (currently 12,5 BTC), which halves approx. every 4 years

A lower bound: *the marginal cost of bitcoin production*

In a competitive commodity market, an agent would undertake mining if the marginal cost per day (electricity consumption) were less than or equal to the marginal product (the number of bitcoins found per day on average multiplied by the dollar price of bitcoin).

Hayes (2015a,b) develops his model by assuming that a miner's daily production of bitcoin depends on its own rate of return, measured in expected bitcoins per day per unit of mining power.

The expected number of bitcoins expected to be produced per day can be calculated as follows:

$$BTC/day^* = [(\beta \cdot \rho)/(\delta \cdot 2^{32})] \cdot sec_{hr} \cdot hr_{day} \qquad (1)$$

where β is the block reward (currently 12,5 BTC/block), ρ is the hashing power employed by a miner, and δ is the difficulty (which is expressed in units of GH/block).

A lower bound: *the marginal cost of bitcoin production*

The constant $\sec_{hr}$ is the number of seconds in an hour (3600), while hr_{day} is the number of hours in a day (24).

The constant 2^{32} relates to the normalized probability of a single hash per second solving a block, and is a feature of the 256-bit encryption at the core of the SHA-256 algorithm.

These constants which normalize the dimensional space for daily time and for the mining algorithm can be summarized by the variable θ, given by $\theta = 24hr_{day} \cdot 3600/2^{32}sec_{hr} =$ 0.000020116567616943. Equation (1) can thus be rewritten compactly as follows:

$$BTC/day^* = \theta \cdot (\beta \cdot \rho)/\delta \tag{2}$$

Hayes (2015a,b) sets $\rho = 1000$ GH/s even though the actual hashing power of a miner is likely to deviate greatly from this value. However, Hayes (2015a,b) argues that this level tends to be a good standard of measure.

A lower bound: *the marginal cost of bitcoin production*

The cost of mining per day, E_{day} can be expressed as follows:

$$E_{day} = (\text{price per kWh} \cdot 24\, hr_{day} \cdot \text{W per GH/s})(\rho/1000\, GH/s) \quad (3)$$

Assuming that the bitcoin market is a competitive market, the marginal product of mining should be equal to its marginal cost, so that the \$/BTC (equilibrium) price level is given by the ratio of (cost/day) / (BTC/day):

$$p^* = E_{day}/(BTC/day^*) \quad (4)$$

$\Rightarrow$ This price level can be though as a price lower bound, below which a miner would operate at a marginal loss and would probably stop mining.

A lower bound: *the marginal cost of bitcoin production*

Example: use the world average electricity cost ≈ 13.5 cents/KWh, the average energy efficiency of bitcoin mining hardware≈ 0.25J/GH

⇒ the average cost per day for a 1000 GH/s mining rig is:

$$\begin{aligned} E_{day} &= (\text{price per kWh} \cdot 24\, hr_{day} \cdot \text{W per GH/s})(\rho/1000\, GH/s) \\ &= (0.135 \cdot 24 \cdot 0.25) \cdot (1,000/1,000) = 0.81\$/day \end{aligned}$$

The number of bitcoins that a 1000 GH/s of mining power can find in a day with a current difficulty of 2227847638504 is equal to

$$\begin{aligned} BTC/day^* &= \theta \cdot (\beta \cdot \rho)/\delta = \\ &= 0.000201165676116943 \cdot (12,5 \cdot 1\mathrm{e}^{12})/2227847638504 \\ &= 0.00011286969561757\, BTC/day. \end{aligned}$$

The \$/BTC price is given by equation (4):

$$\begin{aligned} p^* &= E_{day}/(BTC/day^*) = \\ &= (0.81\$/day)/(0.00011286969561757 BTC/day) \\ &\approx 7176.40\, \$/BTC \end{aligned}$$

A lower bound: *the marginal cost of bitcoin production*

$\Rightarrow$ If we use the efficiency of the best bitcoin mining hardware (Antminer S9) ≈ 0.1 J/GH, then $p^* \approx 2870.56\,\$$

$\Rightarrow$ If we use the efficiency of the best bitcoin mining hardware (Antminer S9) ≈ 0.1 J/GH + Siberian or Chinese electricity costs $\approx$ 3 cents/KWh, then $p^* \approx 637.90\,\$$

A lower bound: *the marginal cost of bitcoin production*

Example 2: use the same approach to compute the lower bound for Bitcoin Cash:

- Using the average energy efficiency of bitcoin mining hardware: $p^*_{BCH} \approx 954.45\,\$$
- Using the efficiency of the best bitcoin mining hardware (Antminer S9) ≈ 0.1 J/GH: $p^*_{BCH} \approx 381.78\,\$$
- Using the efficiency of the best bitcoin mining hardware (Antminer S9) ≈ 0.1 J/GH + Siberian or Chinese electricity costs ≈ 3 cents/KWh: $p^*_{BCH} \approx 84.84\,\$$

Modelling bitcoin price dynamics: VAR & VECM

Most macro-financial analyses devoted to bitcoin prices employ:

1) Vecto-AutoRegression (VAR) models,

$$\Delta \mathbf{Y}_{t-1} = \alpha + \Phi_1 \Delta \mathbf{Y}_{t-1} + \Phi_2 \Delta \mathbf{Y}_{t-2} + \ldots + \Phi_p \Delta \mathbf{Y}_{t-p} + \varepsilon_t \qquad (5)$$

2) Vector Error Correction (VEC) models,

$$\Delta \mathbf{Y}_{t-1} = \alpha + \mathbf{B}\Gamma \mathbf{Y}_{t-1} + \zeta_1 \Delta \mathbf{Y}_{t-1} + \zeta_2 \Delta \mathbf{Y}_{t-2} + \ldots + \zeta_{p-1} \Delta \mathbf{Y}_{t-(p-1)} + \varepsilon_t \qquad (6)$$

where **B** are the factor loadings, while Γ the cointegrating vector.

Kristoufek (2013) is the first author to propose a multivariate approach: hefound a significant bidirectional relationship, where Google trends search queries influence prices and viceversa, suggesting that speculation and trend chasing dominate the bitcoin price dynamics.

Modelling bitcoin price dynamics: VAR & VECM

Glaser et al. (2014) extended previous research by studying the aggregated behavior of new and uninformed Bitcoin users within the time span from 2011 to 2013, to identify why people gather information about Bitcoin and their motivation to subsequently participate in the Bitcoin system.

The main novelty is the use of regressors that are related to both bitcoin **attractiveness** and bitcoin **supply and demand**:

- *daily BTC price data*,
- *daily exchange volumes* in BTC,
- *Bitcoin network volume*, which includes all Bitcoin transfers caused by monetary transactions within the Bitcoin currency network,
- *daily views on the English Bitcoin Wikipedia page* as a proxy for measuring user attention,
- *dummy variables for 24 events gathered from* `https://en.bitcoin.it/wiki/History`.

Modelling bitcoin price dynamics: VAR & VECM

$\rightarrow$ Glaser et al. (2014) are the first to consider both exchange (EV) and network volumes (NV): their idea is that if a customer want to buy bitcoin to pay for goods or services, exchange and network volumes will share similar dynamics, otherwise only exchange-based volumes will be affected.

$\Rightarrow$ They found that the both increases in Wikipedia searches and in exchange volumes do not impact network volumes, and there is no migration between exchange and network volumes, so that they argued that (uninformed) users mostly stay within exchanges, holding Bitcoin only as an alternative investment and not as a currency.

$\Rightarrow$ Glaser et al. (2014) found that Bitcoin users seem to be positively biased towards Bitcoin, because important negative events, like thefts and hacks, did not lead to significant price corrections.

Modelling bitcoin price dynamics: VAR & VECM

Bouoiyour and Selmi (2015), Bouoiyour et al. (2015) and Kancs et al. (2015) are the first studies to consider three sets of drivers to model bitcoin price dynamics:

- **technical drivers (bitcoin supply and demand)**,
- **attractiveness indicators**
- and **macroeconomic variables**.

In general, all papers confirm that bitcoin attractiveness factors are still the main drivers of bitcoin price, followed by traditional supply and demand related variables, while global macro-financial variables play no role.

Example: Bouoiyour and Selmi (2015) use these variables: ...

Modelling bitcoin price dynamics: VAR & VECM

Variable	***Explanation***
	Technical drivers
The exchange-trade ratio (ETR)	Bitcoins are used primarily for two purposes: purchases and exchange rate trading. The Blockchain website provides the total number of transactions and their volume excluding the exchange rate trading. In addition, the ratio between volume of trade (primarily purchases) and exchange transactions is also provided.
Bitcoin monetary velocity (MBV)	It is the frequency at which one unit of bitcoin is used to purchase tradable or non-tradable products for a given period. In the Bitcoin system, the monetary velocity of BitCoin circulation is proxied by the so-called *BitCoin days destroyed*. This variable is calculated by taking the number of BitCoins in transaction and multiplying it by the number of days since those coins were last spent.
The estimated output volume (EOV)	It is similar to the total output volume with the addition of an algorithm which tries to remove change from the total value. This estimate should reflect more accurately the true transaction volume. A negative relationship between the estimated output volume and bitcoin price is expected.
The Hash Rate	The estimated number of giga-hashes per second (billions of hashes per second) the bitcoin network is performing. It is an indicator of the processing power of the Bitcoin network
	Attractiveness indicators
Investors' attractiveness (TTR)	daily Bitcoin views from Google, because it is able to properly depict the speculative character of users
	Macroeconomic variables
The gold price (GP)	Bitcoin does not have an underlying value derived from consumption or production process such as gold.
The Shangai market index (SI)	The Shangai market is considered one of the biggest player in Bitcoin economy and it is considered as a potential source of Bitcoin price volatility.

Modelling bitcoin price dynamics: VAR & VECM

$\Rightarrow$ Using a dataset spanning between 05/12/2010 and 14/06/2014, Bouoiyour and Selmi (2015) found that in the short-run, the investors attractiveness, the exchange-trade ratio, the estimated output volume and the Shangai index have a positive and significantly impact on Bitcoin price, while the monetary velocity, the hash rate and the gold price have no effect.

$\Rightarrow$ Instead, in the long-run, only the exchange-trade ratio and the hash rate have a significant impact on bitcoin price dynamics.

These results hold also with the inclusion of a dummy variable to account for the bankruptcy of a major Chinese bitcoin trading company in 2013, with oil prices, the Dow Jones index and a dummy variable to consider the closure of the Road Silk by the FBI in October 2013.

Modelling bitcoin price dynamics: Bayesian VAR

3) Bayesian VARs models

Bayesian methods treat the value of an unknown model parameter vector θ as a probability distribution $\pi(\theta|Y)$, which is the called the *posterior* distribution of θ given the data Y.

The prior distribution, $\pi(\theta)$, is set externally and reflects the researcher's *prior* ideas on the unknown parameter vector, while $l(Y|\theta)$ is the *likelihood* function, which depends on the information from the given data Y.

The Bayes' theorem is then used to link all these distributions by means of this formula:

$$\pi(\theta|Y) = \frac{\pi(\theta)l(Y|\theta)}{\int \pi(\theta)l(Y|\theta)d\theta}$$

Given that the denominator is a normalizing constant, the posterior is proportional to the product of the likelihood and the prior, that is $\pi(\theta|Y) \propto \pi(\theta)l(Y|\theta)$.

Modelling bitcoin price dynamics: Bayesian VAR

Let consider the following reduced form VAR,

$$Y_t = \Phi_0 + \Phi_1 Y_{t-1} + \ldots + \Phi_p Y_{t-p} + \varepsilon_t, \quad \varepsilon_t \sim N(\mathbf{0}, \Sigma)$$

where $Y_t = (Y_{1t}, \ldots Y_{nt})$ is a $n \times 1$ vector, Φ_0 is a $n \times 1$ vector of constants, Φ_l with $l = 1, \ldots, p$ are the usual autoregression $n \times n$ coefficient matrices.

The previous equation can be written more compactly as $Y_t = \Phi' X_t + \varepsilon_t$ using $X_t = [1 \;\, Y'_{t-1}, \ldots, Y'_{t-p}]'$ and $\Phi = [\Phi_0 \; \Phi_1 \ldots \Phi_p]$. If the variables and shocks are further grouped as follows $Y = [Y_1, \ldots, Y_T]'$, $X = [X_1, \ldots, X_T]'$, $E = [\varepsilon_1, \ldots, \varepsilon_T]'$, we can write the VAR model even more compactly:

$$Y = X\Phi + E$$

A Bayesian VAR combines the likelihood function $L(Y|\Phi, \Sigma)$ with a prior distribution $p(\Phi, \Sigma)$ to get a posterior distribution for the model parameters $p(\Phi, \Sigma|Y)$:

$$p(\Phi, \Sigma|Y) \propto p(\Phi, \Sigma) L(Y|\Phi, \Sigma)$$

Modelling bitcoin price dynamics: Bayesian VAR

There are several possible choices for priors to be used with Bayesian VAR models: I present below the *conjugate normal-inverse Wishart prior* which is a widely used choice and it is implemented into the **bvarr** package.The prior is reported below:

$$\begin{cases} \Sigma \sim \mathcal{IW}(\underline{S}, \underline{\nu}) \\ \Phi|\Sigma \sim N(\underline{\Phi}, \Sigma \otimes \underline{\Omega}) \end{cases}$$

where the scale matrix $\underline{S}$ is diagonal and its non-zero elements assure that the mean of Σ is equal to the fixed covariance matrix of the standard Minnesota prior,

$$(\underline{S})_{ii} = (\underline{\nu} - n - 1)\hat{\sigma}_i^2$$

and σ_i^2 is commonly set to be equal to the variance estimate of residuals in a univariate AR model. The degrees of freedom of the inverse Wishart distribution are set to be greater than or equal to the $\max\{n+2, n+2h-T\}$ to guarantee the existence of the prior variance of the regression parameters and the posterior variances of the forecasts at horizon h.

Modelling bitcoin price dynamics: Bayesian VAR

The matrix $\underline{\Phi}$ is set to $\underline{\Phi} = E(\Phi)$ and the matrices $\underline{\Phi}_l$ are given by:

$$(\underline{\Phi}_l)_{ij} = \begin{cases} \delta_i, \quad i = j, \quad l = 1 \\ 0 \quad \text{otherwise} \end{cases}$$

The matrix $\underline{\Omega}$ is diagonal and it depends on the following hyperparameters:

$$\begin{aligned} \underline{\Omega} &= \text{diag}\{\underline{\Omega}_0, \underline{\Omega}_1, \ldots, \underline{\Omega}_p\} \\ (\underline{\Omega}_l)_{jj} &= \left(\frac{\lambda}{l^{\lambda_l}\hat{\sigma}_j}\right)^2 l = 1, \ldots, p, \quad \underline{\Omega}_0 = \lambda_0^2 \end{aligned}$$

where λ determines the overall tightness of the prior and the relative weight of the prior with respect to the information incorporated in the data, λ_l manages the speed of the decrease of the prior variance with increasing the lag length, while λ_0 controls the relative tightness of the prior for the constant terms.

Modelling bitcoin price dynamics: Bayesian VAR

The posterior distribution formed by combining the previous prior distribution with a likelihood function is also normal - inverse Wishart, see e.g. Zellner (1996):

$$\begin{cases} \Sigma | Y \sim \mathcal{IW}(\overline{S}, \overline{\nu}) \\ \Phi | \Sigma, Y \sim N(\overline{\Phi}, \Sigma \otimes \overline{\Omega}) \end{cases}$$

with the following parameters:

$$\begin{aligned}
\overline{\nu} &= \underline{\nu} + T \\
\overline{\Omega} &= (\underline{\Omega}^{-1} + X'X)^{-1} \\
\overline{\Phi} &= \overline{\Omega} \cdot (\underline{\Omega}^{-1}\underline{\Phi} + X'Y) \\
\overline{S} &= \underline{S} + \hat{E}'\hat{E} + \hat{\Phi}'X'X\hat{\Phi} + \underline{\Phi}'\underline{\Omega}^{-1}\underline{\Phi} - \overline{\Phi}'\overline{\Omega}^{-1}\overline{\Phi} \\
\hat{\Phi} &= (X'X)^{-1}X'Y \\
\hat{E} &= Y - X\hat{\Phi}
\end{aligned}$$

Modelling bitcoin price dynamics: Bayesian VAR

Doan et al. (1984) and Sims (1993) proposed to add two other priors to the previous prior distribution to include the beliefs that the data may be non-stationary and cointegrated:

$\Rightarrow$ A *sum-of-coefficients prior* assumes that the sum of all the lag parameters for each dependent variable is equal to one. This prior is implemented by combining the previous system with the following artificial dummy-observations:

$$
\begin{aligned}
Y^{SC} &= \frac{1}{\lambda_{sc}}[\mathrm{diag}(\delta_1\mu_1, \ldots, \delta_n\mu_n)] \\
X^{SC} &= \frac{1}{\lambda_{sc}}[0_{n\times 1} \quad (1_{1\times p}) \otimes \mathrm{diag}(\delta_1\mu_1, \ldots, \delta_n\mu_n)]
\end{aligned}
$$

where $(1_{1\times p})$ is a unitary $[1 \times p]$ vector, and μ_i is i-th component of vector μ, which contains the average values of initial p observations of all variables in the sample, $\mu = \frac{1}{p}\sum_{t=1}^{p} Y_t$.

When $\lambda_{sc} \to 0$ no cointegration exists and there are as many unit roots as variables.

Modelling bitcoin price dynamics: Bayesian VAR

$\Rightarrow$ The *dummy initial observation prior* proposed by Sims (1993) models the belief that the variables have a common stochastic trend, so that the average value for a variable is a linear combination of the average values of all the other variables.

A single dummy observation is added such that the values of all variables are set to be equal to the averages of the initial conditions μ_i normalized with a scaling factor λ_{io}:

$$
\begin{aligned}
Y^{IO} &= \frac{1}{\lambda_{io}}[(\delta_1\mu_1, \ldots, \delta_n\mu_n)] \\
X^{IO} &= \frac{1}{\lambda_{io}}[1 \quad (1_{1\times p}) \otimes (\delta_1\mu_1, \ldots, \delta_n\mu_n)]
\end{aligned}
$$

When $\lambda_{io} \to 0$, the model assumes that either all variables are stationary with means equal to sample averages of the initial observations, or non-stationary without drift terms and cointegrated.

Modelling bitcoin price dynamics: High-dimensional VAR models with LASSO

The last years have witnessed an increasing statistical literature dealing with the forecasting of high-dimensional multivariate time series, focusing particularly on the *lasso*, see Tibshirani et al. (1996), and its structured variants like the group lasso proposed by Yuan et al. (2006) and the sparse group lasso by Simon et al. (2013).

The R package **BigVAR** adapted the previous penalized regression solution algorithms to a multivariate time series setting: it considers the VARX-L framework proposed by Nicholson et al. (2017) and the class of Hierarchical Vector Autoregression (HVAR) models suggested by Nicholson et al.(2016) that deals with the issue of VAR lag order selection by imposing a nested group lasso penalty.

Given the increasing dimension of cryptocurrencies datasets, these approaches can be of interest to financial professionals and researchers alike. I focus here on HVAR models.

Hierarchical Vector Autoregression (HVAR) models

4) HVAR class of models: Nicholson et al. (2016) proposed a class of models which include the lag order selection into hierarchical group lasso penalties.

HVAR(p) models induce sparsity and a low maximum lag order. Moreover, lag orders are allowed to change across marginal models, that is across variables.

The HVAR penalty structures are reported in Table 1.

Table 1: HVAR penalty functions

Group Name	$\mathcal{P}_Y(\Phi)$
Componentwise	$\sum_{i=1}^{n}\sum_{l=1}^{p}\|\|\Phi_i^{l:p}\|\|_2$
Own/Other	$\sum_{i=1}^{n}\sum_{l=1}^{p}\left[\|\|\Phi_i^{l:p}\|\|_2 + \|\|\Phi_{i,-i}^{l}, \Phi_i^{[l+1]:p}\|\|_2\right]$
Elementwise	$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{p}\|\|\Phi_{ij}^{l:p}\|\|_2$
Lag-weighted Lasso	$\sum_{l=1}^{p} l^{\gamma}\|\|\Phi^{l}\|\|_1$

Hierarchical Vector Autoregression (HVAR) models

The *Componentwise HVAR* penalty allows for the maximum lag order to change across marginal models but, within a single variable equation, all components have the same maximum lag. Therefore, we can have at maximum n different lag orders.

The *Own/Other HVAR* penalty is similar to the Componentwise HVAR, but it prioritizes the coefficients of lagged values of the series of forecasting interest (the so-called 'own' lags) over those of other variables.

$\Rightarrow$ This approach is similar to a Bayesian VAR with a Minnesota Prior (Litterman, 1979) where the variable own lags are considered more informative than the lags of other variables.

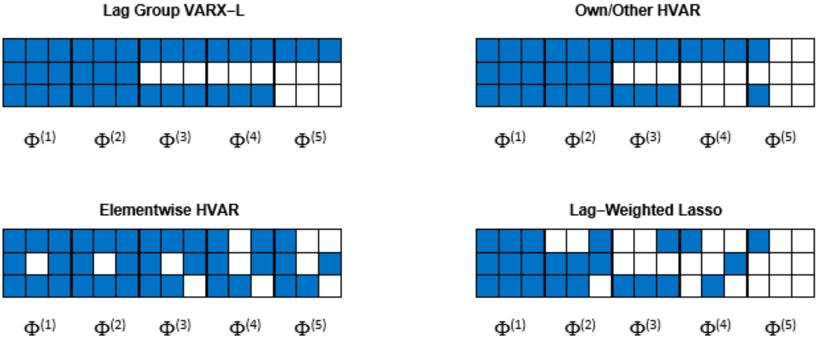
Hierarchical Vector Autoregression (HVAR) models

The *Elementwise HVAR* is the most general structure, because every variable in every equation is allowed to have its own maximum lag so that there can be n^2 possible lag orders.

The *Lag-weighted Lasso* penalty structure is a lasso penalty that increases geometrically with lags and the additional penalty parameter $\gamma \in [0, 1]$ is jointly estimated with λ using sequential cross-validation.

Hierarchical Vector Autoregression (HVAR) models

Examples of the previous four sparsity patterns are reported below:



A (simple) forecasting comparison

I performed a simple exercise to backtest the forecasting performances of the previous multivariate models. I used the dataset `data_bitcoin_multi` from the **bitcoinfinance** package. This is a dataframe of 1447 rows and 12 columns containing the following variables:

- *timestamp*: daily time-stamp;
- *Close*: Average BTCUSD market price across major bitcoin exchanges. Source: blockchain.info;
- *Volume_traded_USD*: The total USD value of trading volume on major bitcoin exchanges. Source: blockchain.info;
- *Google*: Normalized daily Google search data for the word "bitcoin";
- *Transaction_value*: The total estimated value of transactions on the Bitcoin blockchain. Source: blockchain.info;
- *Hash_Rate*: The estimated number of tera hashes per second (trillions of hashes per second) the Bitcoin network is performing. Source: blockchain.info ...

A (simple) forecasting comparison

- *Gold*: Gold price in USD. Source: investing.com;
- *Shanghai_index*: The Shanghai market index. Source: yahoo.finance;
- *total_bitcoins*: The total number of bitcoins that have already been mined; in other words, the current supply of bitcoins on the network. Source: blockchain.info;
- *New_posts*: The number of new posts on online BitCoin forums extracted from bitcointalk.org;
- *New_members*: The number of new members on online BitCoin forums extracted from bitcointalk.org;
- *Dow_Jones*: Dow Jones stock market index. Source: yahoo.finance;

A (simple) forecasting comparison

I use a 250-day rolling window to compute the 1-step and 10-step ahead forecasts for each model, as well as the RMSE and MAE. More specifically, we consider the following models:

- a VAR model with all the variables in levels;
- a VAR model with all the variables in first differences;
- a VAR model with all the variables in log-levels;
- a VAR model with all the variables in first log-differences (= log-returns);
- a VECM model with all the variables in levels/first differences;
- a VECM model with all the variables in log-levels/log-returns;
- a Bayesian VAR model with the conjugate normal-inverse Wishart prior and all the variables in levels;
- a Bayesian VAR model with the conjugate normal-inverse Wishart prior and all the variables in first differences;
- a Bayesian VAR model with the conjugate normal-inverse Wishart prior and all the variables in log-levels;
- a Bayesian VAR model with the conjugate normal-inverse Wishart prior and all the variables in first log-differences (= log-returns);
- a Elementwise HVAR for data in log-returns.

To simplify the computational setting, I considered only multivariate models with lags up to 4 and only one HVAR model.

A (simple) forecasting comparison

	Model	RMSE_one	RMSE_ten	MAE_one	MAE_ten	RMSEr_one	RMSEr_ten
1	VAR	32.90585	6179.76592	15.730572	256.90057	0.09468916	10.8188974
2	VAR_df	33.81572	1730.80027	16.006390	110.41668	0.09093941	3.1583757
3	VAR_ln	26.70353	114.08447	12.700447	56.77481	0.06474223	0.3821900
4	VAR_ln_df	27.28904	103.89433	13.150808	52.27589	0.06501200	0.2269520
5	VECM_df	16.42422	137.75005	4.718925	30.29227	0.10005716	0.7353224
6	VECM_ln_df	25.51986	108.27673	12.780186	55.90240	0.06285254	0.3567914
7	BVAR	26.03046	732.61624	12.533003	84.03390	0.07023995	1.4064061
8	BVAR_df	26.60896	147.43113	13.276180	56.15538	0.06627483	0.7663963
9	BVAR_ln	24.15452	105.64368	11.364833	51.28369	0.05947483	0.2664181
10	BVAR_ln_df	25.68545	98.89014	12.650457	51.93367	0.05938815	0.2051344
11	HVAR	23.79271	98.02010	11.465751	51.43830	0.05763195	0.1993132

	Model	MAEr_one	MAEr_ten	NA_perc
1	VAR	0.05088815	0.5700199	0.00000
2	VAR_df	0.04970261	0.2856069	0.00000
3	VAR_ln	0.03732443	0.1789157	0.00000
4	VAR_ln_df	0.03691062	0.1496810	0.00000
5	VECM_df	0.03540444	0.1608158	44.56613
6	VECM_ln_df	0.03667080	0.1618333	0.00000
7	BVAR	0.03884371	0.2334930	0.00000
8	BVAR_df	0.04024276	0.1693850	0.00000
9	BVAR_ln	0.03199203	0.1459336	0.00000
10	BVAR_ln_df	0.03515411	0.1402560	0.00000
11	HVAR	0.03194456	0.1384539	0.00000

Bitcoin Market Price Discovery

Brandvold et al. (2015) are the first (and so far the only ones) to study the price discovery process in the Bitcoin market, which consists of several independent exchanges.

This topic is frequently discussed in the bitcoin community because knowing which exchange reacts most quickly to new information (thus reflecting the value of Bitcoin most precisely), is clearly of outmost importance for both short-term traders and long-term investors.

The price discovery literature employs mainly three methodologies:

- the information share method by Hasbrouck (1995),
- the permanent-transitory decomposition by Gonzalo and Granger (1995)
- the structural multivariate time series model by de Jong et al. (2001) which is an extension of Harvey (1989).

Bitcoin Market Price Discovery

Brandvold et al. (2015) used the method by de Jong et al. (2001) because

- it has the advantage that the information share is uniquely defined, unlike the information share computed with the Hasbrouck's (1995) model,
- and it takes the variance of innovations into account, unlike Gonzalo and Granger (1995), so that a price series with low innovation variance gets a low information share.

This multivariate model by de Jong et al. (2001) was proposed to estimate the information share of various exchanges with respect to the information generated by the whole market.

Bitcoin Market Price Discovery

⇒ The prices are composed of two components, one common (unobserved) underlying random walk and an idiosyncratic specific noise for each exchange.

⇒ The random walk component is interchangeably referred to either as the efficient price or the fundamental news component.

⇒ It follows immediately from this model structure that the exchanges' prices are cointegrated by construction, while the idiosyncratic component can be due to specific conditions at an exchange, traders' strategic behaviour, or other shocks.

The theoretical setup in Brandvold et al. (2015) assumes n individual exchanges and m corresponding markets, with $m = n$, whereas a market for an exchange is defined as all the other exchanges combined.

Bitcoin Market Price Discovery

Brandvold et al. (2015) denote P^e as the vector of exchange prices, P^m as the vector of market prices, while U^e and U^m represents the vectors of idiosyncratic shocks for the exchanges and the markets, respectively.

P^*denotes the efficient price, p^e= ln P^e, u^e= ln U^e and p^*= ln P^*, so that the logarithm of the n-vector of exchange prices and the m-vector of market prices are given by:

$$\begin{aligned} p_t^e &= p_t^* + u_t^e \\ p_t^m &= p_t^* + u_t^m \end{aligned} \tag{7}$$

where p^* is a random walk.

This is a special case of an unobserved components structural model, see Harvey (1989) for more details.

Bitcoin Market Price Discovery

If we denote the log-returns of the efficient price over the interval $(t-1, t)$ as denoted $r_t = p_t^* - p_{t-1}^*$, then the model assumptions are given below:

$$
\begin{aligned}
& E[r_t^2] = \sigma^2 \\
& E[r_t u_{it}^e] = \psi_i \\
& E[r_t u_{jt}^m] = \psi_j \\
& E[r_t u_{i,t+l}^e] = \gamma_{li}, \quad l \geq 0 \\
& E[r_t u_{j,t+l}^m] = \gamma_{lj}, \quad l \geq 0 \\
& E[r_t u_{i,t-k}^e] = 0, \quad k \geq 0 \\
& E[r_t u_{j,t-k}^m] = 0, \quad k \geq 0 \\
& E[u_{it}^e] = \Omega^e \\
& E[u_{it}^e u_{jt}^m] = \Omega, \quad i = j \\
& E[u_{i,t-k}^e] = 0, \quad k \neq 0 \\
& E[u_{it}^e u_{j,t-k}^m] = 0, \quad k \neq 0
\end{aligned}
\tag{8}
$$

where i refers to exchange i, j to market j, while ψ, γ are $(n \times 1)$ vectors and Ω, Ω^e are $(n \times n)$ matrices.

Bitcoin Market Price Discovery

The fundamental news component r_t can be correlated with concurrent and future idiosyncratic components, but is otherwise uncorrelated.

Instead, the idiosyncratic components are serially uncorrelated and they reflect the noise present in intradaily data.

These restrictions on the correlation structure are needed to identify the model, see Harvey (1989) details. Given the previous structure, the log-returns of observed prices are defined as followed:

$$y_{it} = p_{it} - p_{i,t-1} = p_t^* + u_{it} - p_{t-1}^* - u_{it-1} = r_t + u_{it} - u_{it-1} \quad (9)$$

so that the vectors of exchanges prices and market prices are:

$$\begin{aligned} Y_t^e &= \iota r_t + u_t^e - u_{t-1}^e \\ Y_t^m &= \iota r_t + u_t^m - u_{t-1}^m \end{aligned} \quad (10)$$

where ι is a vector of ones with $n = m$ elements.

Bitcoin Market Price Discovery

Given the assumptions in (8), the serial covariances of Y_t are

$$\begin{aligned} E[Y_t Y_t'] &= \sigma^2 \iota\iota' + \iota\psi' + \psi\iota' + 2\Omega \\ E[Y_t Y_{t-1}'] &= -\psi\iota' - \Omega + \gamma\iota' \\ E[Y_t Y_{t-2}'] &= -\gamma\iota' \end{aligned} \tag{11}$$

Similarly, the serial covariance between an exchange and its corresponding market, that is the covariance between an element in vector Y^e and the corresponding element in vector Y^m , is given by

$$\begin{aligned} E[y_{jt} y_{it}] &= \sigma^2 + \psi_j + \psi_i + 2\omega_{ij} \\ E[y_{jt} y_{i,t-1}] &= -\psi_j - \omega_{ij} + \gamma_j \\ E[y_{jt} y_{i,t-2}] &= -\gamma_j \end{aligned} \tag{12}$$

while the first order autocorrelation for exchanges is

$$\rho_{1,ii} = \frac{-(\psi_i + \omega_{ii}^e - \gamma_i)}{\sigma^2 + 2(\psi_i + \omega_{ii}^e)} \tag{13}$$

Bitcoin Market Price Discovery

The parameter ψ_i -which is the covariance between the fundamental news component and the idiosyncratic component- is of crucial importance because it shows how the market learns after a price change from an individual exchange:

$\Rightarrow$ a high value for ψ_i implies that a price update from that exchange has an high information content for the whole market.

To explain this issue, consider the covariance between the fundamental news component and a price change at an exchange:

$$Cov(y_{it,} r_t) = \sigma^2 + \psi_i \tag{14}$$

which is derived from (8) and (9). It follows immediately that the n covariances between the exchange updates and the fundamental news component are determined by $n+1$ parameters, so that an identifying restriction is needed.

Bitcoin Market Price Discovery

$\Rightarrow$ **de Jong et al. (2001) suggested the idea that the information generated by the price update of each exchange should be equal on overage to σ^2, the variance of r_t .**

Therefore, if we consider the average covariance between the price change of a selected exchange and the fundamental news,

$$\sum_{i=1}^{n} \pi_i Cov(y_{it,} r_t) = \pi'(\sigma^2 \iota + \psi) = \sigma^2 + \pi'\psi \tag{15}$$

where π is a vector of weights adding to one (more later), then the assumption that σ^2 is the unconditional covariance of a exchange price change and the news component imposes $\pi'\psi = 0$.

This restriction is sufficient to identify the model parameters and also leads to a definition of π_i as the *activity share of an exchange*, defined as the *fraction of trades that happened on exchange i*, or simply, *the probability that a trade took place on exchange i* (Brandvold et al., 2015).

Bitcoin Market Price Discovery

⇒ If we multiply the covariance between the fundamental news component and the price change of exchange i -eq. (14)- with the probability π_i , we get a measure of how much information is generated by the price change of exchange i.

⇒ Dividing this by the total information generated in the market σ^2, we obtain the information share for exchange i:

$$IS_i = \frac{(\sigma^2 + \psi_i)\pi_i}{\sigma^2} = \pi_i \left(1 + \frac{\psi_i}{\sigma^2}\right) \tag{16}$$

1. the information shares add to 1, thus simplifying interpretation
2. the joint information share of two exchanges simply equals the sum of their individual information shares.
3. an exchange with a contemporaneous covariance between its idiosyncratic component and the fundamental news component greater than zero $\psi_i > 0$) has a higher information share than activity share.

Bitcoin Market Price Discovery

The estimation procedure of the model parameters consists of two steps:

1. the sample covariances $E[y_{jt} y_{i,t-k}]$, where k=0,1,2, and the autocorrelations $\rho_{1,ii}$ are estimated;
2. the structural parameters are computed using (12)-(13) and a nonlinear optimization softwarer.

$\Rightarrow$ Besides, some parameters can be found directly: given that σ^2 is the variance of r_t and given the assumption by Brandvold et al. (2015) that the seven exchanges in their dataset represent the whole Bitcoin market, σ^2 can be computed as the variance of the aggregated return of the seven exchanges.

$\Rightarrow$ Similarly γ can be computed directly using the sample covariance between the market returns and its corresponding exchange returns lagged two intervals.

$\Rightarrow$ This leaves only ω^e_{ii}, ω_{ij}, ψ_i and ψ_i to be estimated in a 2nd step

Bitcoin Market Price Discovery

The objective function used by Brandvold et al. (2015) to find the remaining parameters with a nonlinear programming solver is given below:

$$Z = \sum_{i=1}^{n} |\pi_i \psi_i| = 0 \tag{17}$$

subject to the following set of constraints

$$\begin{array}{ll}
\rho_{1,ii} = \frac{-(\psi_i + \omega_{ii}^e - \gamma_i)}{\sigma^2 + 2(\psi_i + \omega_{ii}^e)} & (i = 1, \ldots, n) \\
E[y_{jt} y_{i,t-1}] = -\psi_j - \omega_{ij} + \gamma_j & (i = j = 1, \ldots, n) \\
E[y_{jt} y_{i,t-2}] = -\gamma_j & (i = j = 1, \ldots, n) \\
E[y_{it} y_{j,t-2}] = -\gamma_i & (i = j = 1, \ldots, n) \\
\omega_{ii}^e \geq 0 & (i = 1, \ldots, n)
\end{array} \tag{18}$$

Brandvold et al. (2015) tried also alternative objective functions and starting values, but the estimated parameters showed only minimal differences.

Bitcoin Market Price Discovery

Brandvold et al. (2015) highlighted that there is no agreement in the financial literature on how to measure the trading activity of a specific exchange relative to all trading activity in the market (ψ_i).

⇒ They used the simple average of the daily volume and the daily number of trades for each exchange, rescaled so that the sum of the resulting π_i equal to 1.

⇒ Note that the choice of π_i only affects the magnitude of the information share, but not whether ψ_i is positive or negative.

⇒ Brandvold et al. (2015) suggested a simple robustness check: set $\pi_i = 1/n$ for all exchanges and see how the results change.

Bitcoin Market Price Discovery

Brandvold et al. (2015) used data from seven exchanges: Bitfinex, Bitstamp, BTC-e (Btce), BTC China(Btcn) and Mt.Gox (Mtgox), Bitcurex and Canadian Virtual Exchange (Virtex). Data covered the period April 1st 2013–February 25th 2014, till bankruptcy of Mtgox.

They found that the two exchanges with positive ψ for the entire period were Btce and Mtgox, thus indicating that these exchanges were more informative than their competitors.

Similar evidence was provided by the information share, which was highest for Btce and Mtgox (0,322 and 0.366, respectively).

$\Rightarrow$ Information shares change over time: for example, the information share of Btcn first increased from 0.040 in April 2013 to 0.325 in December 2013 because some large Chinese companies (like Baidu) started accepting Bitcoin as payment, but then its information share fell to 0.124 in January 2014 after the Chinese government banned payment companies from clearing Bitcoin.

Bitcoin Market Price Discovery

⇒ An empirical example with R using the function `information_shares()` from the **bitcoinfinance** package.

I show an example using the bitcoin prices from five exchanges covering the time sample [2016-10-20/2017-04-20]: Bitstamp, Itbit, Gdax, Kraken, and Localbitcoins.

The latter is not formally an exchange, but an online service which facilitates over-the-counter trading of local currency for bitcoins, that is it gives the opportunity to a buyer and a seller to conduct direct transactions.

Bitcoin Market Price Discovery

```
data_file<-system.file("extdata", "btcusd_IS.csv", package = "bitcoinFinance")
dat<-read.csv(file = data_file,header = TRUE,sep = ";",dec = ".")

# Vector of activity shares based on trading volumes and trades frequency
pivector<-c(0.33,0.06,0.48,0.11,0.02)
bitcoinFinance::information_shares(dat,pi=pivector, opt_method="nlminb")

  Information shares PSI_coefficients
1         0.24368444    -8.440675e-04
2         0.05667536    -1.788117e-04
3         0.42498757    -3.698466e-04
4         0.11338134     9.919682e-05
5         0.16127130     2.279428e-02

# Robustness check: vector of activity shares set to 1/n for all five exchanges
n<-ncol(dat)-1
pivector<-c(rep(1/n,n))
bitcoinFinance::information_shares(dat,pi=pivector, opt_method="nlminb")

  Information shares PSI_coefficients
1          0.1028536    -0.0015674684
2          0.1074254    -0.0014937014
3          0.1411473    -0.0009495954
4          0.1244755    -0.0012185958
5          0.5240982     0.0052293611
```

Detecting Bubbles and explosive behavior in bitcoin prices

Tests for financial bubbles can be by grouped into two sets:

1. Tests to detect a single bubble:
   - the Log Periodic Power Law (LPPL) model;
   - the Fry (2014) model and the role of volatility.
2. Tests to detect (potentially) multiple bubbles:
   - the DS LPPLS Confidence and Trust indicators;
   - the Generalized-Supremum ADF (GSADF) test;
   - the EXponential Curve Fitting (EXCF) method.

Due to time constraints, I will briefly present only those which can be of interest to financial professionals.

Detecting Bubbles and explosive behavior in bitcoin prices

1) Testing for a single bubble: LPPL models. The expected value of the asset log price in a upward trending bubble according to the LPPL equation is given by,

$$E[\ln p(t)] = A + B(t_c - t)^{\beta} + C(t_c - t)^{\beta} \cos[\omega \ln(t_c - t) - \phi] \quad (19)$$

where $A > 0$ is the value of $[\ln p(t_c)]$ at the critical time t_c which is interpreted as the end of the bubble,

$B < 0$ the increase in $[\ln p(t)]$ over the time unit before the crash

$C \neq 0$ is the proportional magnitude of the oscillations around the exponential growth,

$0 < \beta < 1$ to ensure a finite price at the critical time t_c of the bubble and quantifies the power law acceleration of prices,

ω is the frequency of the oscillations during the bubble,

while $0 < \phi < 2\pi$ is a phase parameter.

Detecting Bubbles and explosive behavior in bitcoin prices

Financial bubbles are defined in the LPPL model as transient regimes of faster-than- exponential price growth resulting from positive feedbacks, and these regimes represent "positive bubbles".

Example: Conditions for a (positive) bubble to occur within this framework:

1. $0 < \beta < 1$, which guarantees that the crash hazard rate accelerates.
2. The second major condition is that the crash rate should be non-negative, as highlighted by van Bothmer and Meister (2003),
$$b \equiv -B\beta - |C|\sqrt{\beta^2 + \omega^2} \geq 0.$$
3. Lin et al. (2014) added a third condition, requiring that the residuals from fitting equation (19) should be stationary.

⇒ MacDonell (2014) used the LPPL model to forecast successfully the bitcoin price crash that took place on December 4, 2013

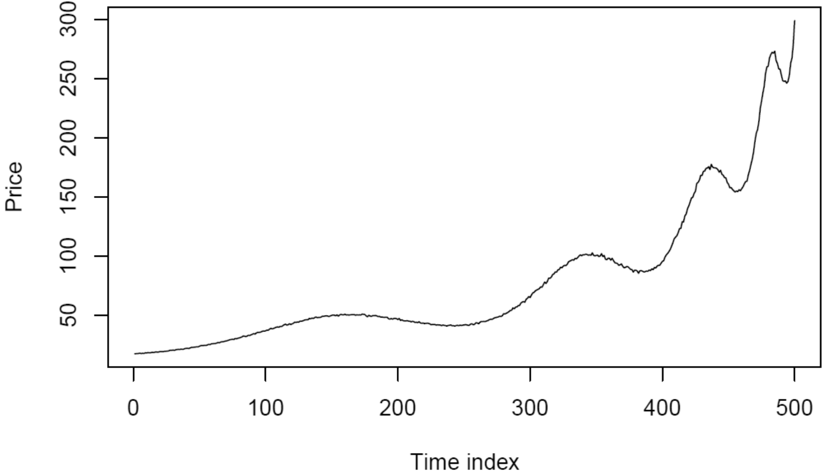
Detecting Bubbles and explosive behavior in bitcoin prices

To have an idea of the LPPL model, let's simulate a price trajectory following this model using the function `lppl_simulate()` from the **bubble** package:

```
lppl_simulate=function(T=500, true_parm){
    bet=true_parm[1]; ome=true_parm[2]; phi=true_parm[3];
    A= true_parm[4]; B =true_parm[5]; C= true_parm[6]; ws=true_parm[7];
    tc=true_parm[8];
    tt_sim=seq(1, T, 1);
    sdum=rep(1,T);
    f_t=(tc - tt_sim)^bet;
    g_t=( (tc - tt_sim)^bet )*cos( ome*log(tc - tt_sim) + phi );
    x=exp(A*sdum +B*f_t + C*g_t +sqrt(ws)*rnorm(T) );
    plot(x, type="l", xlab = "Time index", ylab = "Price")
    return(x)
}
```

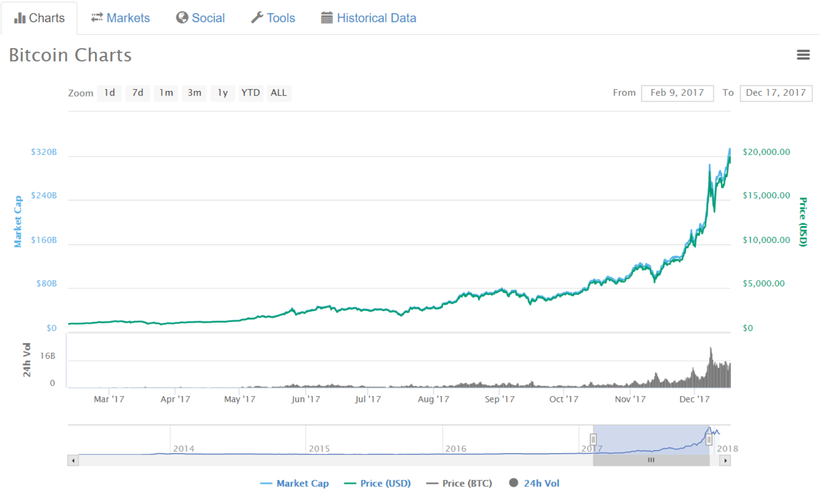
```
tparm=c(0.353689, 9.154368, 2.074608, 7.166421,-0.434324, 0.035405,
        0.000071, 530)
aa=lppl_simulate(500,tparm)
```

Detecting Bubbles and explosive behavior in bitcoin prices



Detecting Bubbles and explosive behavior in bitcoin prices

Sound familiar?



Detecting Bubbles and explosive behavior in bitcoin prices

2) Tests to detect (potentially) multiple bubbles:

The underlying idea of almost all these tests is

- to take a test for a single bubble,
- compute this test using a rolling regression where you change both the starting point and the ending point,
- simulate the distribution for this rolling test, either using the bootstrap for each time window (*LPPL Confidence and Trust indicators*), or simulating it once using the sup statistic (*GSADF test*)

2A) Testing for a multiple bubbles:The DS LPPLS Confidence and Trust indicators.

The starting point is an LPPL model estimated using the procedure proposed by Filimonov and Sornette (2013).

Then, for each fixed end data point t_2, the time series is fitted in shrinking windows (t_1, t_2), whose length $dt = t_2 - t_1$ decreases from 750 trading days to 125 trading days, because the start date t_1 is shifted in steps of five trading days.

The total estimation windows for each t_2 is thus equal to 126.

A set of search space and filter conditions are then imposed to minimize estimation problems. These filters are reported in Table 2, which reproduces Table 1 of Sornette et al. (2015).

Detecting Bubbles and explosive behavior in bitcoin prices

Item	Notation	Search space	Filtering condition 1	Filtering condition 2
Three	β	[0, 2]	[0.01,1.2]	[0.01,0.99]
nonlinear	ω	[1, 50]	[2,25]	[2,25]
parameters	t_c	$[t_2 - 0.2dt,$ $t_2 + 0.2dt]$	$[t_2 - 0.05dt,$ $t_2 + 0.1dt]$	$[t_2 - 0.05dt,$ $t_2 + 0.1dt]$
Oscillations	$\left(\frac{\omega}{2\pi}\right) \ln \left[\frac{\|t_c - t_1\|}{\|t_c - t_2\|}\right]$	-	$[2.5, +\infty)$	$[2.5, +\infty)$
Dumping	$\frac{\beta \|B\|}{\omega \|C\|}$	-	$[0.8, +\infty)$	$[1, +\infty)$
Relative error	$\frac{p_t - \hat{p}_t}{\hat{p}_t}$	-	[0,0.05]	[0,0.2]

Table 2: Search space and filter conditions for the qualification of valid LPPLS fitsa according to Sornette et al. (2015).

Fantazzini (2016) proposed to use the *classical* set of filtering conditions:

- Positive bubble: $0 < \beta < 1, B < 0$,
  $b = [-B \cdot \beta - |C| \cdot \sqrt{\beta^2 + \omega^2}] > 0$ (hazard rate), LPPL residuals stationary at the 5% level (using the KPSS test);
- Negative bubble: $0 < beta < 1, B > 0$,
  $b = [-B \cdot \beta - |C| \cdot \sqrt{\beta^2 + \omega^2}] < 0$ (hazard rate), LPPL residuals stationary at the 5% level (using the KPSS test).

Detecting Bubbles and explosive behavior in bitcoin prices

→ Only the estimated LPPL models that satisfy the previous conditions are considered valid, and this set of qualified fits is then used to build the following two bubble indicators:

- *DS LPPLS confidence*: this indicator represents the fraction of fitting windows for which the estimated LPPL models satisfy the filtering condition 1 reported in Table 2.

It is used to measure the sensitivity of the realized bubble pattern with respect to the time scale *dt* which decreases from 750 to 125 trading days for a total of 126 estimating windows.

A large value shows that the LPPL pattern can be found at most scales and is thus rather strong, whereas a small value signals a weak pattern which is present only in a few estimating windows.

Detecting Bubbles and explosive behavior in bitcoin prices

- *DS LPPLS trust*: it is an indicator that wants to assess the sensitivity of the estimated LPPL models to different realizations of the noise in the financial time series.

The residuals are re-sampled 100 times and they are added to the estimated LPPL model to generate 100 synthetic price time series which are supposed to be independent realizations of the same price pattern.

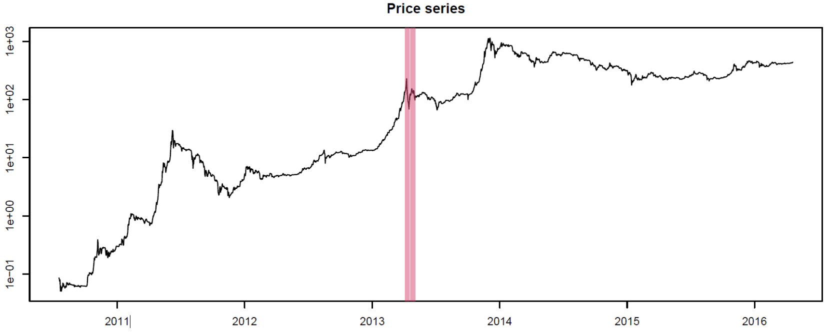
The DS LPPLS trust indicator is then defined as the median level over the 126 time windows of the fraction among the 100 synthetic time series that satisfy the filtering condition 2 in Table 2.

It is a measure of how closely the theoretical LPPL model matches the empirical time series data. Sornette et al. (2015) suggest that, as a rule of thumb, a DS LPPLS trust value larger than 5% shows that the price process is not sustainable and that there is a risk of a critical transition taking place.

Detecting Bubbles and explosive behavior in bitcoin prices

Bitcoin price series with the the DS-LPPLS confidence indicator overlaid using Sornette et al. (2015) filter conditions

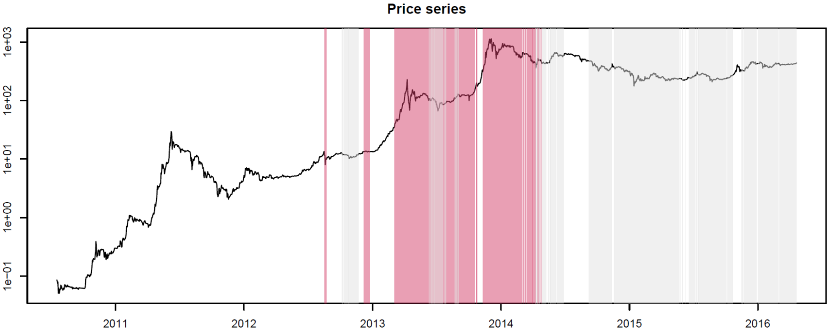
[Warning: extremely simplified computional setting (only 5 estimation windows instead of 126)]



Detecting Bubbles and explosive behavior in bitcoin prices

Bitcoin price series with the the DS-LPPLS confidence indicator overlaid using Fantazzini (2016) filter conditions

[Warning: extremely simplified computional setting (only 5 estimation windows instead of 126)]



Detecting Bubbles and explosive behavior in bitcoin prices

2B) Testing for a multiple bubbles: the Generalized-Supremum ADF test (GSADF).

Tests specifically designed for detecting multiple bubbles were recently proposed by Phillips and Yu (2011), Phillips et al. (2011) and Phillips et al. (2015) and they share the same idea of using sequential tests with rolling estimation windows.

More specifically, *these tests are based on sequential ADF-type regressions* using time windows of different size, and they can consistently identify and date-stamp multiple bubble episodes even in small sample sizes.

We will focus below on the *Generalized-Supremum ADF test (GSADF)* proposed by Phillips, et al. (2015) -PSY henceforward- which builds upon the work by Phillips and Yu (2011) and Phillips et al. (2011), because it has better statistical properties in detecting multiple bubble than the latter two tests.

Detecting Bubbles and explosive behavior in bitcoin prices

This test employs an ADF regression with a rolling sample, where the starting point is given by the fraction r_1 of the total number of observations, the ending point by the fraction r_2, while the window size by $r_w = r_2 - r_1$. The ADF regression is given by

$$y_t = \mu + \rho y_{t-1} + \sum_{i=1}^{p} \phi_{r_w}^{i} \Delta y_{t-i} + \varepsilon_t \tag{20}$$

where the null hypothesis is of a unit root $\rho = 1$ versus an alternative of a mildly explosive autoregressive coefficient $\rho > 1$.

The backward sup ADF test proposed by PSY (2015) fixes the endpoint at r_2 while the window size is expanded from an initial fraction r_0 to r_2, so that the test statistic is given by:

$$BSADF_{r_2}(r_0) = \sup_{r_1 \in [0, r_2 - r_0]} ADF_{r_1}^{r_2} \tag{21}$$

Detecting Bubbles and explosive behavior in bitcoin prices

The generalized sup ADF (GSADF) test is computed by repeatedly performing the BSADF test for each $r_2 \in [r_0, 1]$:

$$GSADF(r_0) = \sup_{r_2 \in [r_0,1]} BSADF_{r_2}(r_0) \tag{22}$$

PSY (2015, Theorem 1) provides the limiting distribution of (22) under the null of a random walk with asymptotically negligible drift (vs an alternative of a mildly explosive process), while critical values are obtained by numerical simulation.

If the null hypothesis of no bubbles is rejected, it is then possible to date-stamp the starting and ending points of one (or more) bubble(s) in a second step. . .

Detecting Bubbles and explosive behavior in bitcoin prices

More specifically,

$\rightarrow$ the *starting point* is given by the date -denoted as T_{r_e}- when the sequence of BSADF test statistics crosses the critical value from below,

$\rightarrow$ whereas the *ending point* -denoted as T_{r_f}- when the BSADF sequence crosses the corresponding critical value from above:

$$\hat{r}_e = \inf_{r_2 \in [r_0, 1]} \left\{ r_2 : BSADF_{r_2}(r_0) > cv_{r_2}^{\beta_T} \right\} \quad (23)$$

$$\hat{r}_f = \inf_{r_2 \in [\hat{r}_e + \delta \log(T)/T, 1]} \left\{ r_2 : BSADF_{r_2}(r_0) < cv_{r_2}^{\beta_T} \right\} \quad (24)$$

where $cv_{r_2}^{\beta_T}$ is the $100(1 - \beta_T)\%$ right-sided critical value of the BSADF statistic based on $\lfloor Tr_2 \rfloor$ observations, $\lfloor \cdot \rfloor$ is the integer fun.

δ is a tuning parameter which determines the minimum duration for a bubble and is usually set to 1, see PSY (2015) and references therein, thus implying a minimum bubble-duration condition of $\ln(T)$ observations.

Detecting Bubbles and explosive behavior in bitcoin prices

Malhotra and Maloo (2014) tested for the presence of multiple bubbles using the GSADF test with data ranging from mid-2011 till February 2014:

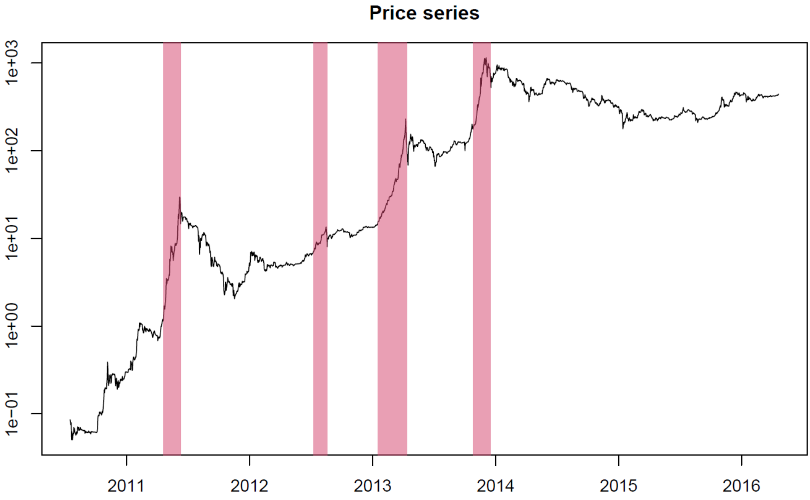
⇒ they found evidence of explosive behaviour in the bitcoin-USD exchange rates during ***August – October 2012*** and ***November, 2013 – February, 2014***.

⇒ They suggested that the first episode of bubble behavior (August – October 2012) could be attributed to the sudden increase in media attention towards bitcoin,

⇒ whereas the second episode to a large set of reasons including the US debt ceiling crisis, the shutdown of Silk Road by the FBI, the rise of Chinese exchange BTC-China, and the increasing number of warnings issued by regulatory authorities and central banks worldwide following the shutdown of the Japanese exchange Mt.Gox.

Detecting Bubbles and explosive behavior in bitcoin prices

Bitcoin price series with periods of explosive behaviour according to the GSADF test highlighted in red, (a minimum bubble duration of 30 days is used).



Conclusions

Financial modelling of cryptocurrencies has only started and there are several possible avenues for further research:

- Econometric methods for market and credit risk management with cryptocurrencies prices are almost non-existent.
- Despite the changes in local regulations, arrival of new investors, police intervention (Silk Road, BTC-e) and massive improvements in mining hardware, there is no research work dealing with structural breaks and long memory.
- All models examined so far are (log-)linear but, considering the behavior of bitcoin prices, nonlinear models could be useful.
- Multi-disciplinary analyses are needed: *IT related papers* focused mainly on electricity costs and energy and computational efficiency, whereas *economic related papers* rarely considered these factors.

Conclusions

More details can be found here:

1. Fantazzini, D., Nigmatullin, E., Sukhanovskaya, V., & Ivliev, S. (2016). Everything you always wanted to know about bitcoin modelling but were afraid to ask. Part 1. *Applied Econometrics*, 44, 5-24. Available at: https://ideas.repec.org/a/ris/apltrx/0301.html

2. Fantazzini, D., Nigmatullin, E., Sukhanovskaya, V., & Ivliev, S. (2017). Everything you always wanted to know about bitcoin modelling but were afraid to ask. Part 2. *Applied Econometrics*, 45, 5-28. Available at: https://ideas.repec.org/a/ris/apltrx/0308.html

3. I am writing a textbook about it . . . stay tuned . . .